# 12 Vector-Valued Functions



# 12.1 Vector-Valued Functions



## Objectives

- Analyze and sketch a space curve given by a vector-valued function.
- Extend the concepts of limits and continuity to vector-valued functions.



## Space Curves and Vector-Valued Functions



## Space Curves and Vector-Valued Functions

A *plane curve* is defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations

$$x = f(t) \text{ and } y = g(t)$$

where $f$ and $g$ are continuous functions of $t$ on an interval $I$.



## Space Curves and Vector-Valued Functions

This definition can be extended naturally to three-dimensional space as follows.

A **space curve** $C$ is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations

$$x = f(t), \quad y = g(t), \quad \text{and} \quad z = h(t)$$

where $f$, $g$ and $h$ are continuous functions of $t$ on an interval $I$.

A new type of function, called a **vector-valued function,** is introduced. This type of function maps real numbers to vectors.

## Space Curves and Vector-Valued Functions

**DEFINITION OF VECTOR-VALUED FUNCTION**

A function of the form

$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ Plane

or

$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ Space

is a **vector-valued function,** where the **component functions** $f$, $g$, and $h$ are real-valued functions of the parameter $t$. Vector-valued functions are sometimes denoted as $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.



## Space Curves and Vector-Valued Functions

Technically, a curve in the plane or in space consists of a collection of points and the defining parametric equations.

Two different curves can have the same graph.

For instance, each of the curves given by

$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ and $\mathbf{r}(t) = \sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}$

has the unit circle as its graph, but these equations do not represent the same curve—because the circle is traced out in different ways on the graphs.



## Space Curves and Vector-Valued Functions

Be sure you see the distinction between the vector-valued function **r** and the real-valued functions *f*, *g*, and *h*.

All are functions of the real variable *t*, but $\mathbf{r}(t)$ is a vector, whereas $f(t)$, $g(t)$, and $h(t)$ are real numbers (for each specific value of *t*).



## Space Curves and Vector-Valued Functions

Vector-valued functions serve dual roles in the representation of curves.

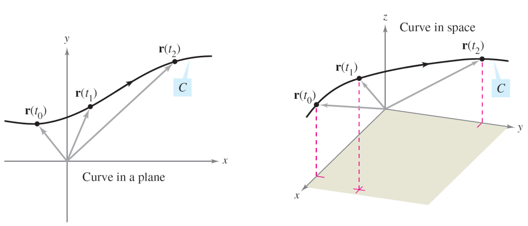
By letting the parameter *t* represent time, you can use a vector-valued function to represent *motion* along a curve.

Or, in the more general case, you can use a vector-valued function to *trace the graph* of a curve.



## Space Curves and Vector-Valued Functions

In either case, the terminal point of the position vector $\mathbf{r}(t)$ coincides with the point $(x, y)$ or $(x, y, z)$ on the curve given by the parametric equations, as shown in Figure 12.1.




Curve $C$ is traced out by the terminal point of position vector $\mathbf{r}(t)$.

**Figure 12.1**



## Space Curves and Vector-Valued Functions

The arrowhead on the curve indicates the curve's *orientation* by pointing in the direction of increasing values of *t*.

Unless stated otherwise, the **domain** of a vector-valued function **r** is considered to be the intersection of the domains of the component functions *f*, *g*, and *h*.

For instance, the domain of $\mathbf{r}(t) = \ln t\,\mathbf{i} + \sqrt{1 - t}\,\mathbf{j} + t\mathbf{k}$ is the interval (0, 1].

## Example 1 – *Sketching a Plane Curve*

Sketch the plane curve represented by the vector-valued function

$\mathbf{r}(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi.$ Vector-valued function

Solution:

From the position vector $\mathbf{r}(t)$, you can write the parametric equations $x = 2\cos t$ and $y = -3\sin t$.

Solving for $\cos t$ and $\sin t$ and using the identity $\cos^2 t + \sin^2 t = 1$ produces the rectangular equation

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1.$$ Rectangular equation



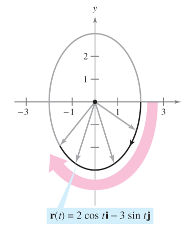
## Example 1 – *Solution*

cont'd

The graph of this rectangular equation is the ellipse shown in Figure 12.2.

The curve has a *clockwise* orientation.

That is, as $t$ increases from 0 to $2\pi$, the position vector $\mathbf{r}(t)$ moves clockwise, and its terminal point traces the ellipse.




The ellipse is traced clockwise as $t$ increases from 0 to $2\pi$.

Figure 12.2



## Limits and Continuity



## Limits and Continuity

**DEFINITION OF THE LIMIT OF A VECTOR-VALUED FUNCTION**

**1.** If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j}$$ Plane

provided $f$ and $g$ have limits as $t \to a$.

**2.** If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$ Space

provided $f$, $g$, and $h$ have limits as $t \to a$.

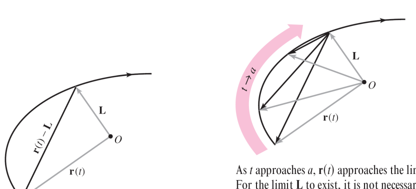


## Limits and Continuity

If $\mathbf{r}(t)$ approaches the vector $\mathbf{L}$ as $t \to a$, the length of the vector $\mathbf{r}(t) - \mathbf{L}$ approaches 0. That is,

$\|\mathbf{r}(t) - \mathbf{L}\| \to 0 \quad$ as $\quad t \to a$.

This is illustrated graphically in Figure 12.6.




As $t$ approaches $a$, $\mathbf{r}(t)$ approaches the limit $\mathbf{L}$. For the limit $\mathbf{L}$ to exist, it is not necessary that $\mathbf{r}(a)$ be defined or that $\mathbf{r}(a)$ be equal to $\mathbf{L}$.

Figure 12.6



## Limits and Continuity

**DEFINITION OF CONTINUITY OF A VECTOR-VALUED FUNCTION**

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** $I$ if it is continuous at every point in the interval.

## Example 5 – *Continuity of Vector-Valued Functions*

Discuss the continuity of the vector-valued function given by

**r**($t$) = $t$**i** + $a$**j** + ($a^2 - t^2$)**k**   *a* is a constant.

at $t$ = 0.

Solution:

As $t$ approaches 0, the limit is

$$\lim_{t\to 0} \mathbf{r}(t) = \left[\lim_{t\to 0} t\right]\mathbf{i} + \left[\lim_{t\to 0} a\right]\mathbf{j} + \left[\lim_{t\to 0} (a^2 - t^2)\right]\mathbf{k}$$

$$= 0\mathbf{i} + a\mathbf{j} + a^2\mathbf{k}$$

$$= a\mathbf{j} + a^2\mathbf{k}.$$



## Example 5 – *Solution*

cont'd

Because

**r**(0) = (0)**i** + ($a$)**j** + ($a^2$)**k**

= $a$**j** + $a^2$**k**

you can conclude that **r** is continuous at $t$ = 0.

By similar reasoning, you can conclude that the vector-valued function **r** is continuous at all real-number values of $t$.



# 12.2 Differentiation and Integration of Vector-Valued Functions





## Objectives

- Differentiate a vector-valued function.
- Integrate a vector-valued function.



## Differentiation of Vector-Valued Functions



## Differentiation of Vector-Valued Functions

The definition of the derivative of a vector-valued function parallels the definition given for real-valued functions.

**DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION**

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all $t$ for which the limit exists. If $\mathbf{r}'(t)$ exists, then **r** is **differentiable at *t*.** If $\mathbf{r}'(t)$ exists for all $t$ in an open interval $I$, then **r** is **differentiable on the interval *I*.** Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

## Differentiation of Vector-Valued Functions

Differentiation of vector-valued functions can be done on a *component-by-component basis*.

To see why this is true, consider the function given by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$



## Differentiation of Vector-Valued Functions

Applying the definition of the derivative produces the following.

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)\mathbf{i} + g(t+\Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \left\{ \left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]\mathbf{i} + \left[\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]\mathbf{j} \right\} \\ &= \left\{ \lim_{\Delta t \to 0} \left[\frac{f(t+\Delta t) - f(t)}{\Delta t}\right]\right\}\mathbf{i} + \left\{ \lim_{\Delta t \to 0} \left[\frac{g(t+\Delta t) - g(t)}{\Delta t}\right]\right\}\mathbf{j} \\ &= f'(t)\mathbf{i} + g'(t)\mathbf{j}\end{aligned}$$

This important result is listed in the theorem 12.1.



## Differentiation of Vector-Valued Functions

Note that the derivative of the vector-valued function **r** is itself a vector-valued function.

You can see from Figure 12.8 that **r′**($t$) is a vector tangent to the curve given by **r**($t$) and pointing in the direction of increasing $t$-values.

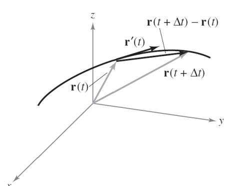



Figure 12.8



## Differentiation of Vector-Valued Functions

**THEOREM 12.1 DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS**

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $f$ and $g$ are differentiable functions of $t$, then
$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}.$$ Plane
2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions of $t$, then
$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$ Space



## Example 1 – *Differentiation of Vector-Valued Functions*

For the vector-valued function given by **r**($t$) = $t$**i** + ($t^2$ + 2)**j**, find **r′**($t$). Then sketch the plane curve represented by **r**($t$), and the graphs of **r**(1) and **r′**(1).

Solution:

Differentiate on a component-by-component basis to obtain

**r′**($t$) = **i** + 2$t$ **j**. Derivative

From the position vector **r**($t$), you can write the parametric equations $x = t$ and $y = t^2 + 2$.

The corresponding rectangular equation is $y = x^2 + 2$. When $t = 1$, **r**(1) = **i** + 3**j** and **r′**(1) = **i** + 2**j**.



## Example 1 – *Solution*

cont'd

In Figure 12.9, **r**(1) is drawn starting at the origin, and **r′**(1) is drawn starting at the terminal point of **r**(1).

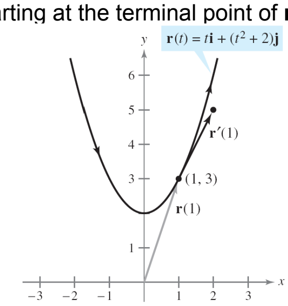



Figure 12.9

## Differentiation of Vector-Valued Functions

The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth on an open interval** $I$ if $f'$, $g'$, and $h'$ are continuous on $I$ and $\mathbf{r}'(t) \neq 0$ for any value of $t$ in the interval $I$.



## Example 3 – *Finding Intervals on Which a Curve Is Smooth*

Find the intervals on which the epicycloid $C$ given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

is smooth.

Solution:

The derivative of $\mathbf{r}$ is

$$\mathbf{r}'(t) = (-5\sin t + 5\sin 5t)\mathbf{i} + (5\cos t - 5\cos 5t)\mathbf{j}.$$

In the interval $[0, 2\pi]$, the only values of $t$ for which

$$\mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j}$$

are $t = 0$, $\pi/2$, $\pi$, $3\pi/2$, and $2\pi$.



## Example 3 – *Solution*

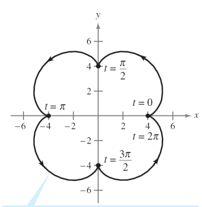
cont'd

Therefore, you can conclude that $C$ is smooth in the intervals $\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$

as shown in Figure 12.10.

In Figure 12.10, note that the curve is not smooth at points at which the curve makes abrupt changes in direction.

Such points are called **cusps** or **nodes.**




$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}$

The epicycloid is not smooth at the points where it intersects the axes.

**Figure 12.10**



## Differentiation of Vector-Valued Functions

**THEOREM 12.2 PROPERTIES OF THE DERIVATIVE**

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of $t$, let $w$ be a differentiable real-valued function of $t$, and let $c$ be a scalar.

1. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$
4. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.



## Example 4 – *Using Properties of the Derivative*

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$$

find

a. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

and

b. $D_t[u(t) \times \mathbf{u}'(t)]$.



## Example 4(a) – *Solution*

Because $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$, you have

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) \\ &\quad + \left(-\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}) \\ &= 2 + 2 + (-1) + \frac{1}{t} \\ &= 3 + \frac{1}{t}. \end{aligned}$$

## Example 4(b) – *Solution*

cont'd

Because $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u}''(t) = 2\mathbf{i}$, you have

$$\begin{aligned} D_t[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0} \\ &= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix}\mathbf{k} \\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k} \\ &= 2\mathbf{j} + 4t\mathbf{k}. \end{aligned}$$



# Integration of Vector-Valued Functions



## Integration of Vector-Valued Functions

The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

**DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS**

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $f$ and $g$ are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} \qquad \text{Plane}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k} \qquad \text{Space}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$



## Integration of Vector-Valued Functions

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector **C**.

For instance, if $\mathbf{r}(t)$ is a three-dimensional vector-valued function, then for the indefinite integral $\int \mathbf{r}(t)\,dt$, you obtain three constants of integration

$$\int f(t)\,dt = F(t) + C_1, \qquad \int g(t)\,dt = G(t) + C_2, \qquad \int h(t)\,dt = H(t) + C_3$$

where $F'(t) = f(t)$, $G'(t) = g(t)$, and $H'(t) = h(t)$.

These three *scalar* constants produce one *vector* constant of integration,

$\int \mathbf{r}(t)\,dt = [F(t) + C_1]\mathbf{i} + [G(t) + C_2]\mathbf{j} + [H(t) + C_3]\mathbf{k}$



## Integration of Vector-Valued Functions

$= [F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}] + [C_1\mathbf{i} + C_2\,\mathbf{j} + C_3\mathbf{k}]$

$= \mathbf{R}(t) + \mathbf{C}$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.



## Example 5 – *Integrating a Vector-Valued Function*

Find the indefinite integral $\int (t\,\mathbf{i} + 3\mathbf{j})\,dt$.

Solution:

Integrating on a component-by-component basis produces

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt = \frac{t^2}{2}\mathbf{i} + 3t\mathbf{j} + \mathbf{C}.$$

## 12.3 Velocity and Acceleration





## Objectives

- Describe the velocity and acceleration associated with a vector-valued function.
- Use a vector-valued function to analyze projectile motion.



## Velocity and Acceleration



## Velocity and Acceleration

As an object moves along a curve in the plane, the coordinates $x$ and $y$ of its center of mass are each functions of time $t$.

Rather than using the letters $f$ and $g$ to represent these two functions, it is convenient to write $x = x(t)$ and $y = y(t)$.

So, the position vector $\mathbf{r}(t)$ takes the form

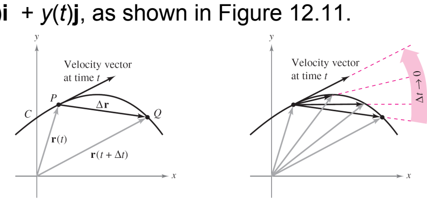
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}. \quad \text{Position vector}$$



## Velocity and Acceleration

To find the velocity and acceleration vectors at a given time $t$, consider a point $Q(x(t + \Delta t), y(t + \Delta t))$ that is approaching the point $P(x(t), y(t))$ along the curve $C$ given by

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, as shown in Figure 12.11.




As $\Delta t \to 0$, $\frac{\Delta \mathbf{r}}{\Delta t}$ approaches the velocity vector.

**Figure 12.11**



## Velocity and Acceleration

As $\Delta t \to 0$, the direction of the vector $\overrightarrow{PQ}$ (denoted by $\Delta \mathbf{r}$) approaches the *direction of motion* at time $t$.

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

$$\frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

If this limit exists, it is defined as the **velocity vector** or **tangent vector** to the curve at point $P$.

## Velocity and Acceleration

Note that this is the same limit used to define **r**′(*t*). So, the direction of **r**′(*t*) gives the direction of motion at time *t*.

Moreover, the magnitude of the vector **r**′(*t*)

$$\|\mathbf{r}'(t)\| = \|x'(t)\mathbf{i} + y'(t)\mathbf{j}\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

gives the **speed** of the object at time *t*.

Similarly, you can use **r**″(*t*) to find acceleration.



## Velocity and Acceleration

**DEFINITIONS OF VELOCITY AND ACCELERATION**

If $x$ and $y$ are twice-differentiable functions of $t$, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time $t$ are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$$

$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$



## Velocity and Acceleration

For motion along a space curve, the definitions are similar.

That is, if **r**(*t*) = *x*(*t*)**i** + *y*(*t*)**j** + *z*(*t*)**k**, *you* have

**Velocity** = **v**(*t*) = **r**′(*t*) = x′(*t*)**i** + *y*′(*t*)**j** + *z*′(*t*)**k**

**Acceleration** = **a**(*t*) = **r**″(*t*) = *x*″(*t*)**i** + *y*″(*t*)**j** + *z*″(*t*)**k**

**Speed** = $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$



## Example 1 – *Finding Velocity and Acceleration Along a Plane Curve*

Find the velocity vector, speed, and acceleration vector of a particle that moves along the plane curve *C* described by

$$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}.$$ Position vector

Solution:

The velocity vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}.$$ Velocity vector

The speed (at any time) is

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2}} = 1.$$ Speed



## Example 1 – *Solution*

cont'd

The acceleration vector is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{1}{2}\sin\frac{t}{2}\mathbf{i} - \frac{1}{2}\cos\frac{t}{2}\mathbf{j}.$$ Acceleration vector



## Projectile Motion

# Projectile Motion

You now have the machinery to derive the parametric equations for the path of a projectile.

Assume that gravity is the only force acting on the projectile after it is launched. So, the motion occurs in a vertical plane, which can be represented by the *xy*-coordinate system with the origin as a point on Earth's surface, as shown in Figure 12.17.

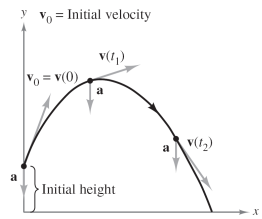



**Figure 12.17**



# Projectile Motion

For a projectile of mass *m,* the force due to gravity is

$\mathbf{F} = -mg\mathbf{j}$ Force due to gravity

where the acceleration due to gravity is
$g$ = 32 feet per second per second, or
9.81 meters per second per second.

By **Newton's Second Law of Motion,** this same force produces an acceleration $\mathbf{a} = \mathbf{a}(t)$, and satisfies the equation $\mathbf{F} = m\mathbf{a}$.

Consequently, the acceleration of the projectile is given by $m\mathbf{a} = -mg\mathbf{j}$, which implies that

$\mathbf{a} = -g\mathbf{j}$. Acceleration of projectile



# Example 5 – *Derivation of the Position Function for a Projectile*

A projectile of mass *m* is launched from an initial position $\mathbf{r}_0$ with an initial velocity $\mathbf{v}_0$. Find its position vector as a function of time.

Solution:

Begin with the acceleration $\mathbf{a}(t) = -g\mathbf{j}$ and integrate twice.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int -g\mathbf{j}\,dt = -gt\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (-gt\mathbf{j} + \mathbf{C}_1)dt = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$$



# Example 5 – *Solution*

cont'd

You can use the facts that $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$ to solve for the constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$.

Doing this produces $\mathbf{C}_1 = \mathbf{v}_0$ and $\mathbf{C}_2 = \mathbf{r}_0$.

Therefore, the position vector is

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0.$$ Position vector



# Projectile Motion

In many projectile problems, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not given explicitly.

Often you are given the initial height $h$, the initial speed $v_0$ and the angle $\theta$ at which the projectile is launched, as shown in Figure 12.18.

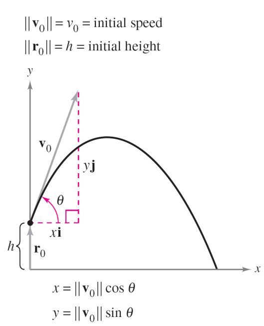



**Figure 12.18**



# Projectile Motion

From the given height, you can deduce that $\mathbf{r}_0 = h\mathbf{j}$. Because the speed gives the magnitude of the initial velocity, it follows that $v_0 = \|\mathbf{v}_0\|$ and you can write

$$\mathbf{v}_0 = x\mathbf{i} + y\mathbf{j}$$
$$= (\|\mathbf{v}_0\|\cos\theta)\mathbf{i} + (\|\mathbf{v}_0\|\sin\theta)\mathbf{j}$$
$$= v_0\cos\theta\mathbf{i} + v_0\sin\theta\mathbf{j}.$$

## Projectile Motion

So, the position vector can be written in the form

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0$$ Position vector

$$= -\frac{1}{2}gt^2\mathbf{j} + tv_0\cos\theta\mathbf{i} + tv_0\sin\theta\mathbf{j} + h\mathbf{j}$$

$$= (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}.$$



## Projectile Motion

**THEOREM 12.3 POSITION FUNCTION FOR A PROJECTILE**

Neglecting air resistance, the path of a projectile launched from an initial height $h$ with initial speed $v_0$ and angle of elevation $\theta$ is described by the vector function

$$\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where $g$ is the acceleration due to gravity.



## Example 6 – *Describing the Path of a Baseball*

A baseball is hit 3 feet above ground level at 100 feet per second and at an angle of 45° with respect to the ground, as shown in Figure 12.19. Find the maximum height reached by the baseball. Will it clear a 10-foot-high fence located 300 feet from home plate?

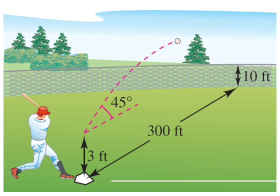



Figure 12.19



## Example 6 – *Solution*

You are given $h = 3$, and $v_0 = 100$, and $\theta = 45°$.

So, using $g = 32$ feet per second per second produces

$$\mathbf{r}(t) = \left(100\cos\frac{\pi}{4}\right)t\mathbf{i} + \left[3 + \left(100\sin\frac{\pi}{4}\right)t - 16t^2\right]\mathbf{j}$$

$$= \left(50\sqrt{2}\,t\right)\mathbf{i} + \left(3 + 50\sqrt{2}\,t - 16t^2\right)\mathbf{j}$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = 50\sqrt{2}\,\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}.$$



## Example 6 – *Solution*

cont'd

The maximum height occurs when

$$y'(t) = 50\sqrt{2} - 32t = 0$$

which implies that

$$t = \frac{25\sqrt{2}}{16} \approx 2.21 \text{ seconds.}$$

So, the maximum height reached by the ball is

$$y = 3 + 50\sqrt{2}\left(\frac{25\sqrt{2}}{16}\right) - 16\left(\frac{25\sqrt{2}}{16}\right)^2 = \frac{649}{8} \approx 81 \text{ feet.}$$ Maximum height when $t \approx 2.21$ seconds



## Example 6 – *Solution*

cont'd

The ball is 300 feet from where it was hit when

$$300 = x(t) = 50\sqrt{2}\,t.$$

Solving this equation for $t$ produces $t = 3\sqrt{2} \approx 4.24$ seconds.

At this time, the height of the ball is

$$y = 3 + 50\sqrt{2}\left(3\sqrt{2}\right) - 16\left(3\sqrt{2}\right)^2$$

$= 303 - 288$

$= 15$ feet. Height when $t \approx 4.24$ seconds

Therefore, the ball clears the 10-foot fence for a home run.

# 12.4 Tangent Vectors and Normal Vectors





## Objectives

- Find a unit tangent vector at a point on a space curve.
- Find the tangential and normal components of acceleration.



## Tangent Vectors and Normal Vectors



## Tangent Vectors and Normal Vectors

**DEFINITION OF UNIT TANGENT VECTOR**

Let $C$ be a smooth curve represented by **r** on an open interval $I$. The **unit tangent vector** $\mathbf{T}(t)$ at $t$ is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{r}'(t) \neq \mathbf{0}.$$



## Example 1 – *Finding the Unit Tangent Vector*

Find the unit tangent vector to the curve given by

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

when $t = 1$.

Solution:

The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}. \qquad \text{Derivative of } \mathbf{r}(t)$$

So, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \text{Definition of } \mathbf{T}(t)$$

$$= \frac{1}{\sqrt{1 + 4t^2}}(\mathbf{i} + 2t\mathbf{j}). \qquad \text{Substitute for } \mathbf{r}'(t).$$



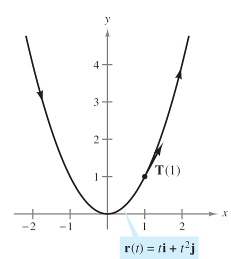
## Example 1 – *Solution*

cont'd

When $t = 1$, the unit tangent vector is

$$\mathbf{T}(1) = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

as shown in Figure 12.20.




The direction of the unit tangent vector depends on the orientation of the curve.

**Figure 12.20**

## Tangent Vectors and Normal Vectors

The **tangent line to a curve** at a point is the line that passes through the point and is parallel to the unit tangent vector.



## Example 2 – *Finding the Tangent Line at a Point on a Curve*

Find **T**($t$) and then find a set of parametric equations for the tangent line to the helix given by

$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$

at the point $\left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$.

Solution:

The derivative of **r**($t$) is $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \mathbf{k}$, which implies that $\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$.

Therefore, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$



## Example 2 – *Solution* cont'd

$$= \frac{1}{\sqrt{5}}(-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \mathbf{k}). \quad \text{Unit tangent vector}$$

At the point $(\sqrt{2}, \sqrt{2}, \pi/4)$, $t = \pi/4$ and the unit tangent vector is

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{5}}\left(-2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}\right)$$

$$= \frac{1}{\sqrt{5}}(-\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + \mathbf{k}).$$



## Example 2 – *Solution* cont'd

Using the direction numbers $a = -\sqrt{2}$, $b = \sqrt{2}$, and $c = 1$, and the point $(x_1, y_1, z_1) = (\sqrt{2}, \sqrt{2}, \pi/4)$, you can obtain the following parametric equations (given with parameter $s$).

$$x = x_1 + as = \sqrt{2} - \sqrt{2}s$$

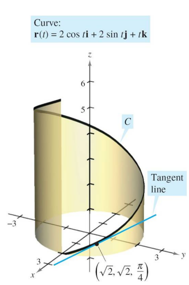
$$y = y_1 + bs = \sqrt{2} + \sqrt{2}s$$

$$z = z_1 + cs = \frac{\pi}{4} + s$$



## Example 2 – *Solution* cont'd

This tangent line is shown in Figure 12.21.




The tangent line to a curve at a point is determined by the unit tangent vector at the point.

**Figure 12.21**



## Tangent Vectors and Normal Vectors

In Example 2, there are infinitely many vectors that are orthogonal to the tangent vector **T**($t$). One of these is the vector **T**′($t$) . This follows the property

$\mathbf{T}(t) \cdot \mathbf{T}(t) = \|\mathbf{T}(t)\|^2 = 1 \Rightarrow \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$

By normalizing the vector **T**′($t$) , you obtain a special vector called the **principal unit normal vector,** as indicated in the following definition.

**DEFINITION OF PRINCIPAL UNIT NORMAL VECTOR**

Let $C$ be a smooth curve represented by **r** on an open interval $I$. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** at $t$ is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

## Example 3 – *Finding the Principle Unit Normal Vector*

Find **N**($t$) and **N**(1) for the curve represented by

**r**($t$) = $3t\mathbf{i} + 2t^2\mathbf{j}$.

Solution:

By differentiating, you obtain

$$\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{9 + 16t^2}$$

which implies that the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}}(3\mathbf{i} + 4t\mathbf{j}). \qquad \text{Unit tangent vector}$$



## Example 3 – *Solution*

cont'd

Using Theorem 12.2, differentiate **T**($t$) with respect to $t$ to obtain

$$\mathbf{T}'(t) = \frac{1}{\sqrt{9 + 16t^2}}(4\mathbf{j}) - \frac{16t}{(9 + 16t^2)^{3/2}}(3\mathbf{i} + 4t\mathbf{j})$$

$$= \frac{12}{(9 + 16t^2)^{3/2}}(-4t\mathbf{i} + 3\mathbf{j})$$

$$\|\mathbf{T}'(t)\| = 12\sqrt{\frac{9 + 16t^2}{(9 + 16t^2)^3}} = \frac{12}{9 + 16t^2}.$$



## Example 3 – *Solution*

cont'd

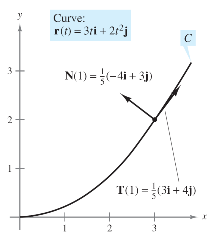
Therefore, the principal unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}}(-4t\mathbf{i} + 3\mathbf{j}). \qquad \text{Principal unit normal vector}$$

When $t = 1$, the principal unit normal vector is

$$\mathbf{N}(1) = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$$

as shown in Figure 12.22.




The principal unit normal vector points toward the concave side of the curve.

**Figure 12.22**



## Tangent Vectors and Normal Vectors

The principal unit normal vector can be difficult to evaluate algebraically. For plane curves, you can simplify the algebra by finding

$$\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad \text{Unit tangent vector}$$

and observing that **N**($t$) must be either

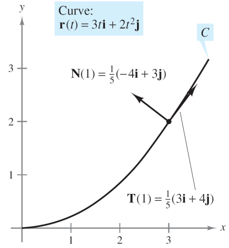
$$\mathbf{N}_1(t) = y(t)\mathbf{i} - x(t)\mathbf{j} \quad \text{or} \quad \mathbf{N}_2(t) = -y(t)\mathbf{i} + x(t)\mathbf{j}.$$



## Tangent Vectors and Normal Vectors

Because $\sqrt{[x(t)]^2 + [y(t)]^2} = 1$, it follows that both $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$ are unit normal vectors.

The *principal* unit normal vector **N** is the one that points toward the concave side of the curve, as shown in Figure 12.22



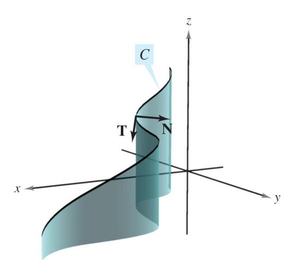

The principal unit normal vector points toward the concave side of the curve.

**Figure 12.22**



## Tangent Vectors and Normal Vectors

This also holds for curves in space. That is, for an object moving along a curve $C$ in space, the vector **T**($t$) points in the direction the object is moving, whereas the vector **N**($t$) is orthogonal to **T**($t$) and points in the direction in which the object is turning, as shown in Figure 12.23.




At any point on a curve, a unit normal vector is orthogonal to the unit tangent vector. The *principal* unit normal vector points in the direction in which the curve is turning.

**Figure 12.23**

## Tangential and Normal Components of Acceleration





**THEOREM 12.4 ACCELERATION VECTOR**

If $\mathbf{r}(t)$ is the position vector for a smooth curve $C$ and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

The coefficients of **T** and **N** in the proof of Theorem 12.4 are called the **tangential and normal components of acceleration** and are denoted by $a_\mathbf{T} = D_t[\|\mathbf{v}\|]$ and $a_\mathbf{N} = \|\mathbf{v}\| \; \|\mathbf{T}'\|$.

So, you can write

$$\mathbf{a}(t) = a_\mathbf{T}\mathbf{T}(t) + a_\mathbf{N}\mathbf{N}(t).$$





The following theorem gives some convenient formulas for $a_\mathbf{N}$ and $a_\mathbf{T}$.

**THEOREM 12.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION**

If $\mathbf{r}(t)$ is the position vector for a smooth curve $C$ [for which $\mathbf{N}(t)$ exists], then the tangential and normal components of acceleration are as follows.

$$a_\mathbf{T} = D_t[\|\mathbf{v}\|] = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_\mathbf{N} = \|\mathbf{v}\| \, \|\mathbf{T}'\| = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_\mathbf{T}^2}$$

Note that $a_\mathbf{N} \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration.**





Find the tangential and normal components of acceleration for the position vector given by $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$ .

Solution:

Begin by finding the velocity, speed, and acceleration.

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} - \mathbf{j} + 2t\mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{9 + 1 + 4t^2} = \sqrt{10 + 4t^2}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{k}$$





By Theorem 12.5, the tangential component of acceleration is

$$a_\mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{4t}{\sqrt{10 + 4t^2}}$$ Tangential component of acceleration

and because

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 6\mathbf{j}$$

the normal component of acceleration is

$$a_\mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{4 + 36}}{\sqrt{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}.$$ Normal component of acceleration



## 12.5 Arc Length and Curvature

## Objectives

- Find the arc length of a space curve.
- Use the arc length parameter to describe a plane curve or space curve.
- Find the curvature of a curve at a point on the curve.
- Use a vector-valued function to find frictional force.



## Arc Length



## Arc Length

**THEOREM 12.6 ARC LENGTH OF A SPACE CURVE**

If $C$ is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, on an interval $[a, b]$, then the arc length of $C$ on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

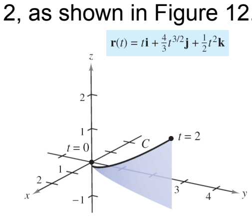


## Example 1 – *Finding the Arc Length of a Curve in Space*

Find the arc length of the curve given by

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

from $t = 0$ to $t = 2$, as shown in Figure 12.28.




As $t$ increases from 0 to 2, the vector $\mathbf{r}(t)$ traces out a curve.

**Figure 12.28**



## Example 1 – *Solution*

Using $x(t) = t$, $y(t) = \frac{4}{3}t^{3/2}$, and $z(t) = \frac{1}{2}t^2$, you obtain $x'(t) = 1$, $y'(t) = 2t^{1/2}$, and $z'(t) = t$.

So, the arc length from $t = 0$ and $t = 2$ is given by

$$s = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \qquad \text{Formula for arc length}$$

$$= \int_0^2 \sqrt{1 + 4t + t^2}\, dt$$

$$= \int_0^2 \sqrt{(t+2)^2 - 3}\, dt \qquad \text{Integration tables (Appendix B), Formula 26}$$

$$= \left[\frac{t+2}{2}\sqrt{(t+2)^2 - 3} - \frac{3}{2}\ln\left|(t+2) + \sqrt{(t+2)^2 - 3}\right|\right]_0^2$$

$$= 2\sqrt{13} - \frac{3}{2}\ln\left(4 + \sqrt{13}\right) - 1 + \frac{3}{2}\ln 3 \approx 4.816.$$



## Arc Length Parameter

## Arc Length Parameter

**DEFINITION OF ARC LENGTH FUNCTION**

Let $C$ be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$. For $a \le t \le b$, the **arc length function** is given by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du.$$

The arc length $s$ is called the **arc length parameter.** (See Figure 12.30.)

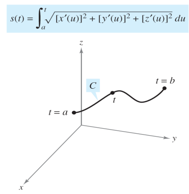



Figure 12.30



## Arc Length Parameter

Using the definition of the arc length function and the Second Fundamental Theorem of Calculus, you can conclude that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$ Derivative of arc length function

In differential form, you can write

$$ds = \|\mathbf{r}'(t)\|\, dt.$$

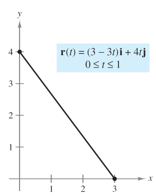


## Example 3 – *Finding the Arc Length Function for a Line*

Find the arc length function $s(t)$ for the line segment given by

$\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\,\mathbf{j}, \quad 0 \le t \le 1$

and write $\mathbf{r}$ as a function of the parameter $s$. (See Figure 12.31.)




The line segment from (3, 0) to (0, 4) can be parametrized using the arc length parameter $s$.

Figure 12.31



## Example 3 – *Solution*

Because $\mathbf{r}'(t) = -3\mathbf{i} + 4\mathbf{j}$ and

$$\|\mathbf{r}'(t)\| = \sqrt{(-3)^2 + 4^2} = 5$$

you have

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du$$

$$= \int_0^t 5\, du$$

$$= 5t.$$



## Example 3 – *Solution* cont'd

Using $s = 5t$ (or $t = s/5$), you can rewrite $\mathbf{r}$ using the arc length parameter as follows.

$$\mathbf{r}(s) = \left(3 - \tfrac{3}{5}s\right)\mathbf{i} + \tfrac{4}{5}s\,\mathbf{j}, \quad 0 \le s \le 5$$



## Arc Length Parameter

**THEOREM 12.7 ARC LENGTH PARAMETER**

If $C$ is a smooth curve given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \quad \text{or} \quad \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$$

where $s$ is the arc length parameter, then

$$\|\mathbf{r}'(s)\| = 1.$$

Moreover, if $t$ is *any* parameter for the vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}'(t)\| = 1$, then $t$ must be the arc length parameter.

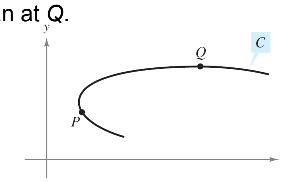
# Curvature



## Curvature

An important use of the arc length parameter is to find **curvature**—the measure of how sharply a curve bends.

For instance, in Figure 12.32 the curve bends more sharply at *P* than at *Q*, and you can say that the curvature is greater at *P* than at *Q*.

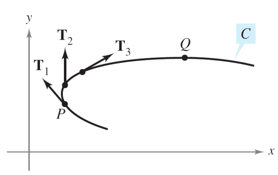



Curvature at *P* is greater than at *Q*.
**Figure 12.32**



## Curvature

You can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector **T** with respect to the arc length *s*, as shown in Figure 12.33.




The magnitude of the rate of change of **T** with respect to the arc length is the curvature of a curve.
**Figure 12.33**



## Curvature

**DEFINITION OF CURVATURE**

Let *C* be a smooth curve (in the plane *or* in space) given by $\mathbf{r}(s)$, where *s* is the arc length parameter. The **curvature** *K* at *s* is given by

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|.$$

A circle has the same curvature at any point. Moreover, the curvature and the radius of the circle are inversely related. That is, a circle with a large radius has a small curvature, and a circle with a small radius has a large curvature.

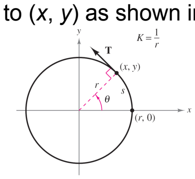


## Example 4 – *Finding the Curvature of a Circle*

Show that the curvature of a circle of radius *r* is $K = 1/r$.

Solution:
Without loss of generality you can consider the circle to be centered at the origin.

Let $(x, y)$ be any point on the circle and let *s* be the length of the arc from $(r, 0)$ to $(x, y)$ as shown in Figure 12.34.




The curvature of a circle is constant.
**Figure 12.34**



## Example 4 – *Solution*

cont'd

By letting $\theta$ be the central angle of the circle, you can represent the circle by

$$\mathbf{r}(\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}. \qquad \theta \text{ is the parameter.}$$

Using the formula for the length of a circular arc $s = r\theta$, you can rewrite $\mathbf{r}(\theta)$ in terms of the arc length parameter as follows.

$$\mathbf{r}(s) = r\cos\frac{s}{r}\,\mathbf{i} + r\sin\frac{s}{r}\,\mathbf{j} \qquad \text{Arc length } s \text{ is the parameter.}$$

## Example 4 – *Solution*

cont'd

So, $\mathbf{r}'(s) = -\sin\frac{s}{r}\mathbf{i} + \cos\frac{s}{r}\mathbf{j}$, and it follows that $\|\mathbf{r}'(s)\| = 1$, which implies that the unit tangent vector is

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = -\sin\frac{s}{r}\mathbf{i} + \cos\frac{s}{r}\mathbf{j}$$

and the curvature is given by

$$K = \|\mathbf{T}'(s)\| = \left\| -\frac{1}{r}\cos\frac{s}{r}\mathbf{i} - \frac{1}{r}\sin\frac{s}{r}\mathbf{j} \right\| = \frac{1}{r}$$

at every point on the circle.



## Curvature

**THEOREM 12.8 FORMULAS FOR CURVATURE**

If $C$ is a smooth curve given by $\mathbf{r}(t)$, then the curvature $K$ of $C$ at $t$ is given by

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Because $\|\mathbf{r}'(t)\| = ds/dt$ , the first formula implies that curvature is the ratio of the rate of change in the tangent vector **T** to the rate of change in arc length. To see that this is reasonable, let $\Delta t$ be a "small number." Then,

$$\frac{\mathbf{T}'(t)}{ds/dt} \approx \frac{[\mathbf{T}(t+\Delta t) - \mathbf{T}(t)]/\Delta t}{[s(t+\Delta t) - s(t)]/\Delta t} = \frac{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)}{s(t+\Delta t) - s(t)} = \frac{\Delta\mathbf{T}}{\Delta s}.$$



## Curvature

In other words, for a given $\Delta s$, the greater the length of $\Delta\mathbf{T}$, the more the curve bends at $t$, as shown in Figure 12.35.

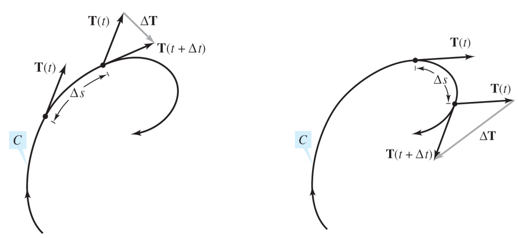



**Figure 12.35**



## Example 5 – *Finding the Curvature of a Space Curve*

Find the curvature of the curve given by $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$.

Solution:

It is not apparent whether this parameter represents arc length, so you should use the formula $K = \|\mathbf{T}'(t)\|/\|\mathbf{r}'(t)\|$.

$\mathbf{r}'(t) = 2\mathbf{i} + 2t\,\mathbf{j} - t^2\mathbf{k}$

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2 \qquad \text{Length of } \mathbf{r}'(t)$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}}{t^2 + 2}$$

$$\mathbf{T}'(t) = \frac{(t^2+2)(2\mathbf{j} - 2t\mathbf{k}) - (2t)(2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k})}{(t^2+2)^2}$$



## Example 5 – *Solution*

cont'd

$$= \frac{-4t\mathbf{i} + (4 - 2t^2)\mathbf{j} - 4t\mathbf{k}}{(t^2+2)^2}$$

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2 + 16 - 16t^2 + 4t^4 + 16t^2}}{(t^2+2)^2}$$

$$= \frac{2(t^2+2)}{(t^2+2)^2}$$

$$= \frac{2}{t^2+2} \qquad \text{Length of } \mathbf{T}'(t)$$

Therefore,

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{(t^2+2)^2}. \qquad \text{Curvature}$$



## Curvature

**THEOREM 12.9 CURVATURE IN RECTANGULAR COORDINATES**

If $C$ is the graph of a twice-differentiable function given by $y = f(x)$, then the curvature $K$ at the point $(x, y)$ is given by

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

## Curvature

Let $C$ be a curve with curvature $K$ at point $P$. The circle passing through point $P$ with radius $r = 1/K$ is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point $P$.

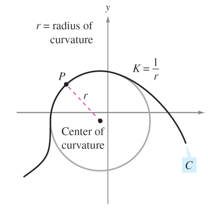
The radius is called the **radius of curvature** at $P$, and the center of the circle is called the **center of curvature.**



## Curvature

The circle of curvature gives you the curvature $K$ at a point $P$ on a curve. Using a compass, you can sketch a circle that lies against the concave side of the curve at point $P$, as shown in Figure 12.36.

If the circle has a radius of $r$, you can estimate the curvature to be $K = 1/r$.




The circle of curvature

**Figure 12.36**



## Example 6 – *Finding Curvature in Rectangular Coordinates*

Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$. Sketch the circle of curvature at (2, 1).

Solution:

The curvature at $x = 2$ is as follows.

$$y' = 1 - \frac{x}{2} \qquad y' = 0$$

$$y'' = -\frac{1}{2} \qquad y'' = -\frac{1}{2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} \qquad K = \frac{1}{2}$$



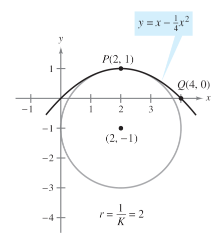
## Example 6 – *Solution*

cont'd

Because the curvature at $P(2, 1)$ is $\frac{1}{2}$, it follows that the radius of the circle of curvature at that point is 2.

So, the center of curvature is (2, –1) as shown in Figure 12.37.

[In the figure, note that the curve has the greatest curvature at $P$.]




The circle of curvature

**Figure 12.37**



## Curvature

**THEOREM 12.10 ACCELERATION, SPEED, AND CURVATURE**

If $\mathbf{r}(t)$ is the position vector for a smooth curve $C$, then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

where $K$ is the curvature of $C$ and $ds/dt$ is the speed.



## Example 7 – *Tangential and Normal Components of Acceleration*

Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for the curve given by

$$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\mathbf{j} - \tfrac{1}{3}t^3\mathbf{k}.$$

Solution:

From Example 5, you know that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = t^2 + 2 \quad \text{and} \quad K = \frac{2}{(t^2 + 2)^2}.$$

Therefore,

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 2t \qquad \text{Tangential component}$$

and

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{(t^2 + 2)^2}(t^2 + 2)^2 = 2. \qquad \text{Normal component}$$

# Application



# Application

There are many applications in physics and engineering dynamics that involve the relationships among speed, arc length, curvature, and acceleration. One such application concerns frictional force.

A moving object with mass $m$ is in contact with a stationary object. The total force required to produce an acceleration **a** along a given path is

$$\mathbf{F} = m\mathbf{a} = m\left(\frac{d^2s}{dt^2}\right)\mathbf{T} + mK\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

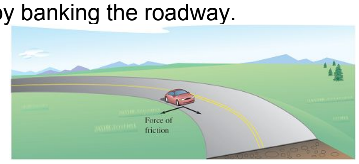
$$= ma_{\mathbf{T}}\mathbf{T} + ma_{\mathbf{N}}\mathbf{N}.$$

The portion of this total force that is supplied by the stationary object is called the **force of friction.**



# Application

For example, if a car moving with constant speed is rounding a turn, the roadway exerts a frictional force that keeps the car from sliding off the road. If the car is not sliding, the frictional force is perpendicular to the direction of motion and has magnitude equal to the normal component of acceleration, as shown in Figure 12.39. The potential frictional force of a road around a turn can be increased by banking the roadway.




The force of friction is perpendicular to the direction of motion.

**Figure 12.39**



# Example 8 – *Frictional Force*

A 360-kilogram go-cart is driven at a speed of 60 kilometers per hour around a circular racetrack of radius 12 meters, as shown in Figure 12.40. To keep the cart from skidding off course, what frictional force must the track surface exert on the tires?

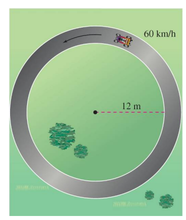



**Figure 12.40**



# Example 8 – *Solution*

The frictional force must equal the mass times the normal component of acceleration.

For this circular path, you know that the curvature is

$$K = \frac{1}{12}. \qquad \text{Curvature of circular racetrack}$$

Therefore, the frictional force is

$$ma_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2$$

$$= (360 \text{ kg})\left(\frac{1}{12 \text{ m}}\right)\left(\frac{60{,}000 \text{ m}}{3600 \text{ sec}}\right)^2$$

$$\approx 8333 \text{ (kg)(m)/sec}^2.$$



# Application

**SUMMARY OF VELOCITY, ACCELERATION, AND CURVATURE**

| | | |
|---|---|---|
| | Let $C$ be a curve (in the plane or in space) given by the position function | |
| | $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ | Curve in the plane |
| | $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$ | Curve in space |
| **Velocity vector, speed, and acceleration vector:** | $\mathbf{v}(t) = \mathbf{r}'(t)$ | Velocity vector |
| | $\lVert\mathbf{v}(t)\rVert = \frac{ds}{dt} = \lVert\mathbf{r}'(t)\rVert$ | Speed |
| | $\mathbf{a}(t) = \mathbf{r}''(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$ | Acceleration vector |
| **Unit tangent vector and principal unit normal vector:** | $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\lVert\mathbf{r}'(t)\rVert}$ and $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\lVert\mathbf{T}'(t)\rVert}$ | |

# Application

cont'd

| | | |
|---|---|---|
| **Unit tangent vector and principal unit normal vector:** | $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\\|\mathbf{r}'(t)\\|}$ and $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\\|\mathbf{T}'(t)\\|}$ | |
| **Components of acceleration:** | $a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \dfrac{\mathbf{v} \cdot \mathbf{a}}{\\|\mathbf{v}\\|} = \dfrac{d^2s}{dt^2}$ <br> $a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \dfrac{\\|\mathbf{v} \times \mathbf{a}\\|}{\\|\mathbf{v}\\|} = \sqrt{\\|\mathbf{a}\\|^2 - a_{\mathbf{T}}^2} = K\left(\dfrac{ds}{dt}\right)^2$ | |
| **Formulas for curvature in the plane:** | $K = \dfrac{\|y''\|}{[1 + (y')^2]^{3/2}}$ | $C$ given by $y = f(x)$ |
| | $K = \dfrac{\|x'y'' - y'x''\|}{[(x')^2 + (y')^2]^{3/2}}$ | $C$ given by $x = x(t)$, $y = y(t)$ |
| **Formulas for curvature in the plane or in space:** | $K = \\|\mathbf{T}'(s)\\| = \\|\mathbf{r}''(s)\\|$ | $s$ is arc length parameter. |
| | $K = \dfrac{\\|\mathbf{T}'(t)\\|}{\\|\mathbf{r}'(t)\\|} = \dfrac{\\|\mathbf{r}'(t) \times \mathbf{r}''(t)\\|}{\\|\mathbf{r}'(t)\\|^3}$ | $t$ is general parameter. |
| | $K = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\\|\mathbf{v}(t)\\|^2}$ | |
| | Cross product formulas apply only to curves in space. | |