Paul's Online Notes

Section 6-8 : Tangent, Normal And Binormal Vectors

In this section we want to look at an application of derivatives for vector functions. Actually, there are a couple of applications, but they all come back to needing the first one.

In the past we've used the fact that the derivative of a function was the slope of the tangent line. With vector functions we get exactly the same result, with one exception.

Given the vector function, $\vec{r}\,(t)$, we call $\vec{r}\,'(t)$ the **tangent vector** provided it exists and provided $\vec{r}\,'(t) \neq \vec{0}$. The tangent line to $\vec{r}\,(t)$ at P is then the line that passes through the point P and is parallel to the tangent vector, $\vec{r}\,'(t)$. Note that we really do need to require $\vec{r}\,'(t) \neq \vec{0}$ in order to have a tangent vector. If we had

$$\vec{r}\,'(t) = \vec{0}$$

we would have a vector that had no magnitude and so couldn't give us the direction of the tangent.

Also, provided $\vec{r}\,'(t) \neq \vec{0}$, the **unit tangent vector** to the curve is given by,

$$\vec{T}\,(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$$

While, the components of the unit tangent vector can be somewhat messy on occasion there are times when we will need to use the unit tangent vector instead of the tangent vector.

Example 1 Find the general formula for the tangent vector and unit tangent vector to the curve given by $\vec{r}\,(t) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$.

Hide Solution

First, by general formula we mean that we won't be plugging in a specific t and so we will be finding a formula that we can use at a later date if we'd like to find the tangent at any point on the curve. With that said there really isn't all that much to do at this point other than to do the work.

Here is the tangent vector to the curve.

$$\vec r\,'(t) = 2t\,\vec i + 2\cos t\,\vec j - 2\sin t\,\vec k$$

To get the unit tangent vector we need the length of the tangent vector.

$$\begin{aligned}\left\|\vec r\,'(t)\right\| &= \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t}\\ &= \sqrt{4t^2+4}\end{aligned}$$

The unit tangent vector is then,

$$\begin{aligned}\vec T(t) &= \frac{1}{\sqrt{4t^2+4}}\left(2t\,\vec i + 2\cos\, t\,\vec j - 2\sin t\,\vec k\right)\\ &= \frac{2t}{\sqrt{4t^2+4}}\,\vec i + \frac{2\cos t}{\sqrt{4t^2+4}}\,\vec j - \frac{2\sin t}{\sqrt{4t^2+4}}\,\vec k\end{aligned}$$

Example 2 Find the vector equation of the tangent line to the curve given by $\vec r(t) = t^2\,\vec i + 2\sin t\,\vec j + 2\cos t\,\vec k$ at $t = \dfrac{\pi}{3}$.

Hide Solution

First, we need the tangent vector and since this is the function we were working with in the previous example we can just reuse the tangent vector from that example and plug in $t = \frac{\pi}{3}$.

$$\vec r\,'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\,\vec i + 2\cos\left(\frac{\pi}{3}\right)\vec j - 2\sin\left(\frac{\pi}{3}\right)\vec k = \frac{2\pi}{3}\,\vec i + \vec j - \sqrt{3}\,\vec k$$

We'll also need the point on the line at $t = \frac{\pi}{3}$ so,

$$\vec r\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\,\vec i + \sqrt{3}\,\vec j + \vec k$$

The vector equation of the line is then,

$$\vec r(t) = \left\langle \frac{\pi^2}{9}, \sqrt{3}, 1\right\rangle + t\left\langle \frac{2\pi}{3}, 1, -\sqrt{3}\right\rangle$$

Before moving on let's note a couple of things about the previous example. First, we could have used the unit tangent vector had we wanted to for the parallel vector. However, that would have made for a more complicated equation for the tangent line.

Second, notice that we used $\vec r(t)$ to represent the tangent line despite the fact that we used that as well for the function. Do not get excited about that. The $\vec r(t)$ here is much like y is with normal functions. With normal functions, y is the generic letter that we used to represent functions and $\vec r(t)$ tends to be used in the same way with vector functions.

Next, we need to talk about the **unit normal** and the **binormal** vectors.

The unit normal vector is defined to be,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

The unit normal is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well. We've already seen normal vectors when we were dealing with **Equations of Planes**. They will show up with some regularity in several Calculus III topics.

The definition of the unit normal vector always seems a little mysterious when you first see it. It follows directly from the following fact.

Fact

Suppose that $\vec{r}(t)$ is a vector such that $\|\vec{r}(t)\| = c$ for all t. Then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

To prove this fact is pretty simple. From the fact statement and the relationship between the magnitude of a vector and the dot product we have the following.

$$\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2 \qquad \text{for all} t$$

Now, because this is true for all t we can see that,

$$\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(c^2) = 0$$

Also, recalling the fact from the previous section about differentiating a dot product we see that,

$$\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}'(t) \cdot \vec{r}(t)$$

Or, upon putting all this together we get,

$$2\vec{r}'(t) \cdot \vec{r}(t) = 0 \qquad \Rightarrow \qquad \vec{r}'(t) \cdot \vec{r}(t) = 0$$

Therefore $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

The definition of the unit normal then falls directly from this. Because $\vec{T}(t)$ is a unit vector we know that $\left\|\vec{T}(t)\right\| = 1$ for all t and hence by the Fact $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. However, because $\vec{T}(t)$ is tangent to the curve, $\vec{T}'(t)$ must be orthogonal, or normal, to the curve as well and so be a normal vector for the curve. All we need to do then is divide by $\left\|\vec{T}'(t)\right\|$ to arrive at a unit normal vector.

Next, is the binormal vector. The binormal vector is defined to be,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.

Example 3 Find the normal and binormal vectors for $\vec{r}(t) = \langle t, 3\sin t, 3\cos t\rangle$.

Hide Solution

We first need the unit tangent vector so first get the tangent vector and its magnitude.

$$\begin{aligned} \vec{r}'(t) &= \langle 1, 3\cos t, -3\sin t\rangle \\ \left\|\vec{r}'(t)\right\| &= \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10} \end{aligned}$$

The unit tangent vector is then,

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude.

$$\begin{aligned} \vec{T}'(t) &= \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle \\ \left\|\vec{T}'(t)\right\| &= \sqrt{\frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} \end{aligned}$$

The unit normal vector is then,

$$\vec{N}(t) = \frac{\sqrt{10}}{3}\left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle = \langle 0, -\sin t, -\cos t\rangle$$

Finally, the binormal vector is,

$$\begin{aligned} \vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\ &= \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & -\frac{3}{\sqrt{10}}\sin t & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t \\ 0 & -\sin t & -\cos t & 0 & -\sin t \end{array} \right. \\ &= -\frac{3}{\sqrt{10}}\cos^2 t\,\vec{i} - \frac{1}{\sqrt{10}}\sin t\,\vec{k} + \frac{1}{\sqrt{10}}\cos t\,\vec{j} - \frac{3}{\sqrt{10}}\sin^2 t\,\vec{i} \\ &= -\frac{3}{\sqrt{10}}\,\vec{i} + \frac{1}{\sqrt{10}}\cos t\,\vec{j} - \frac{1}{\sqrt{10}}\sin t\,\vec{k} \end{aligned}$$

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